L3 Moments, Deviation and Tail Inequalities

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Some contents of this handout are adapted from Prof. Yitong Yin's handout on Concentration of Measure.

1 Moments and Deviations

Definition (Moment). The kth moment of a random variable X is $\mathbb{E}[X^k]$.

Definition (Central Moment). The kth central moment of a random variable X is $\mathbb{E}[(X-\mathbb{E}X)^k]$.

2 Markov's Inequality

Theorem (Markov's Inequality). Let X be a random variable that assumes only nonnegative values. Then, for all $a>0$,

$$\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

3 Chebyshev's Inequality

Theorem (Chebyshev's Inequality). For any $a>0$,

$$\Pr(|X-\mathbb{E}[X]| \geq a) \leq \frac{\operatorname{Var}[X]}{a^2}$$

4 The Chernoff Bound

4.1 Moment Generating Function

Definition (Moment Generating Function). The moment generating function of a random variable X is defined as $\mathbf{E}\left[\mathrm{e}^{\lambda X}\right]$ where λ is the parameter of the function.

4.2 The Chernoff Bound

Definition (Poisson trials) Poisson trials are independent random variables $X_1, X_2, \ldots$ where each X_i only takes on value 0 or 1.

Theorem (Chernoff bound: the upper tail)

Let $X=\sum_{i=1}^n X_i$, where $X_1, X_2, \ldots, X_n$ are independent Poisson trials. Let $\mu=\mathbf{E}[X]$.

Then for any $\delta>0$,

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu .$$

Theorem (Chernoff bound: the lower tail)

Let $X = \sum_{i=1}^{n} X_i$, where $X_1, X_2, \ldots, X_n$ are independent Poisson trials. Let $\mu = \mathbf{E}[X]$.
Then for any $0 < \delta < 1$,

$$\Pr[X \le (1-\delta)\mu] \le \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}.$$

Theorem (Chernoff bound: some useful forms)

Let $X = \sum_{i=1}^{n} X_i$, where $X_1, X_2, \ldots, X_n$ are independent Poisson trials. Let $\mu = \mathbf{E}[X]$. Then

1. for any $0 < \delta < 1$,

$$\begin{aligned} \Pr[X \ge (1+\delta)\mu] &< \exp\left(-\frac{\mu\delta^2}{3}\right); \\ \Pr[X \le (1-\delta)\mu] &< \exp\left(-\frac{\mu\delta^2}{2}\right); \end{aligned} \tag{1}$$

2. for $t \ge 2e\mu$,

$$\Pr[X \ge t] \le 2^{-t}. \tag{2}$$

5 Further Reading

1. *Probability and Computing*
   (a) Ch3 Moments and Deviations
   See 3.1~3.3 for Markov's inequality and Chebyshev's inequality

   (b) Ch4 Chernoff and Hoeffding Bounds
   See 4.2 and 4.3 for detailed proofs of Chernoff bounds and its various forms;
   See 4.5 for the Hoeffding Bound
   See 4.4, 4.6 for some applications of Chernoff and Hoeffding Bounds
   See Ex 4.21 for the steps to prove the runtime of randomized quicksort is $O(n \log n)$ w.h.p

   (c) Ch5 Balls, Bins, and Random Graphs
   See 5.1, 5.2 for the balls into bins model

2. *Randomized Algorithms*
   (a) Ch3 Moments and Deviations

   (b) Ch4 Tail Inequalities

3. *Concentration of Measure for the Analysis of Randomized Algorithms*
   (a) Ch1 Chernoff–Hoeffding Bounds

(b) Ch2 Applications of the Chernoff–Hoeffding Bounds
See 2.3 for detailed analysis of the search cost of skip list
See 2.4 for detailed analysis of the runtime of randomized quicksort is $O(n \log n)$ w.h.p

4. Slides and handouts from Prof. Yitong Yin

   (a) Moment and Deviation from *Probability and Mathematical Statistics*

   (b) Concentration of Measure from *Probability and Mathematical Statistics*

   (c) slides and handouts of *Concentration of Measure* from *Advanced Algorithms*

5. Lecture notes from other professors

   (a) Quick Sort with High Probability By *Sariel Har-Peled*

   (b) Lectures 18, 19, 20 & 21 (Randomized Algorithms & High Probability Bounds) from CSE 548: Analysis of Algorithms by Rezaul A. Chowdhury

   (c) Exercises of Chapter 4 by *Chuang-Chieh Lin*
   You can find the answer to the Ex 4.21 of *Probability and Computing*