Prerequisite for L2

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1 Notation

Notation	Meaning
A	a set
Ω	sample space (the universe)
$A \setminus B$	$\{x : x \in A \wedge x \notin B\}$
A^c	the compliment of A, *i.e.*, $\Omega \setminus A$
Σ	the set of events
$\mathcal{F}$	a collection of set / a set family
$\Pr$	the probability measure

2 Definition

The occurrence or non-occurrence of a random event depends upon the chain of circumstances involved, which is called an **experiment** or **trial**; the result of an experiment is called its **outcome**.

Definition (Sample Space):

The set of all possible outcomes of an experiment is called the **sample space** and is denoted by Ω.

Definition (σ-field/σ-algebra):

A collection $\mathcal{F}$ of subsets of Ω is called a σ-field if it satisfies the following conditions:

1. $\varnothing \in \mathcal{F}$
2. If $A_1, A_2, \cdots \in \mathcal{F}$, then $\bigcup_{i=0}^{\infty} A_i \in \mathcal{F}$
3. if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

We think of ***events*** as subsets of the sample space Ω. We require the set of events Σ is a σ-field.

Events A and B are called **disjoint** if their intersection is the empty set $\varnothing$;

- $\varnothing$ is called the **impossible event**.
- The set Ω called the **certain event**.

Note

The power set of Ω, which is written 2^Ω, is obviously a σ-field. However, when Ω is infinite, its power set is too large a collection for probabilities to be assigned reasonably to all its members.

Definition (Probability Measure):

A **probability measure** Pr on (Ω, Σ) is a function $\text{Pr} : \Sigma \to [0, 1]$ satisfying

1. $P(\varnothing) = 0, P(\Omega) = 1$.
2. if $A_1, A_2, \ldots$ is a collection of disjoint members of Σ, in that $A_i \cap A_j = \varnothing$ for all pairs i, j satisfying $i \neq j$, then

$$\text{Pr}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \text{Pr}(A_i)$$

An event A is called **null** if $\text{Pr}(A) = 0$. If $\text{Pr}(A) = 1$, we say that A occurs **almost surely**.

Definition (Probability Space):

The triple $(\Omega, \Sigma, \text{Pr})$, comprising a set Ω, a σ-field Σ of subsets of Ω, and a probability measure Pr on (Ω, Σ), is called a **probability space**.

Definition (Conditional Probability of Event)

If $\text{Pr}(B) > 0$ then the **conditional probability** that A occurs given that B occurs is defined to

$$\text{Pr}(A \mid B) = \frac{\text{Pr}(A \cap B)}{\text{Pr}(B)}$$

pronounced 'the probability of A given B', or sometimes 'the probability of A conditioned (or conditional) on B'.

Definition (Independence of Events)

Events A and B are called **independent** if

$$\text{Pr}(A \cap B) = \text{Pr}(A)\,\text{Pr}(B)$$

More generally, a family $\{A_i : i \in I\}$ is called **independent** if

$$\text{Pr}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \text{Pr}(A_i)$$

for any *finite* subsets J of I.

A family $\{A_i : i \in I\}$ is called **k-wise independent** if

$$\Pr\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \Pr(A_i)$$

for any *finite* subsets J of I such that $|J| = k$. **2-wise independent** is also called **pairwise independent**.

Definition (Random Variable)

A random variable is a function $X : \Omega \to \mathbb{R}$ with the property that $\{\omega \in \Omega : X(\omega) \in \Sigma\}$ for each $x \in \mathbb{R}$. Such a function is said to be **Σ-measurable**.

3 Theorem

Basic Lemma derived from Axioms

1. $\Pr(A^c) = 1 - \Pr(A)$
2. If $A \subseteq B$ then $\Pr(B) \geq \Pr(A)$
3. $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
4. (**Inclusion-Exclusion Principle / Poincaré's Formula**)

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|-1} \Pr\left(\bigcap_{i \in I} A_i\right).$$

Theorem (Law of Total Probability)

let $B_1, B_2, \ldots, B_n$ be a partition of Ω such that $\Pr(B_i) > 0$ for all i. Then

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A \mid B_i) \Pr(B_i)$$

Theorem (Boole's Inequalities / Union Bounds)

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \Pr(A_i)$$

Theorem (Bayes's Formula)

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \Pr(A)}{\Pr(B)}$$